—Chapter 7—

The Statistical Growth of Asset Portfolios

When you're first thinking through an idea, it's important not to get bogged down in complexity. Thinking simply and clearly is hard to do.
Richard Branson

Truth is to be found in the simplicity, and not in the multiplicity and confusion of things.
Isaac Newton

The concept of a general equilibrium has no relevance to the real world (in other words, classical economics is an exercise in futility).
George Soros

The teacher who is indeed wise does not bid you to enter the house of his wisdom but rather leads you to the threshold of your mind.
Klalil Gibran

7.1 Introduction[1]

This chapter considers the nature of asset growth that is subject to normally distributed compounding growth rates. As such, the model represents a straightforward development of Chapter 6. Nevertheless, the mathematical development is able to shed light on the economy – and the market of asset claims on the economy – as an organic process of growth and decline; the small firm size effect in the Fama and French three-factor (FF-3F) model; as well as observations that portfolios of stocks with high individual volatility appear to outperform the market.

The chapter is arranged as follows. Section 7.2 considers the case for a normal distribution as an acceptable model of asset price formation. Section 7.3 considers the essential mathematical implications of normally distributed growth rates, before Section 7.4 adapts the findings for asset price formation. Section 7.5 progresses to consider the implications of normally distributed continuously compounding growth rates for the small

[1] This chapter develops ideas that were presented in the journal *Financial Analyst Journal* (Dempsey, 2002). I am indeed grateful and indebted to the editors and reviewers of this journal for supporting and publishing this work notwithstanding its unorthodoxy.

firm size effect in the FF-3F model, before a final section, Section 7.6, concludes with a discussion[2].

7.2 Normally Distributed Growth Rates as a Foundation for Asset Price Formation

When the philosopher Réne Descartes wished to build his philosophy of knowledge on a foundation that could not be disputed, he arrived, by a process of continually questioning backwards from any supposition he made, at *I think therefore I am*. The very act of doubting one's own existence, he argued, serves to establish the reality of one's own existence, or at least of one's thought. In seeking the same for finance, we are prompted to ask how we might commence an understanding of asset price formation and growth from such a reliable foundation.

In nature, the growth response, of an organism's population (of flies, or locusts, or bacteria, etc.) does not hold still for a month or a year, and then review the past to determine how it must respond. Rather, the process of growth and decline is continuous in occurring from the position of that organism's ever-updated population.

Such continuously compounding growth was captured for our bank deposit in Chapter 6.4 (p. 78) by applying a growth rate x% per period as (x/100)/N over many N sub-intervals, which with large N was identified as the growth factor e^x per period. However, such continuously compounding growth fails to capture organic growth in one most essential manner: Whereas the bank deposit growth was assumed to be riskless, in nature, very little is riskless. The exponential growth of our organism *can vary*.

Even in chaos, however, nature typically enforces a form of stability. Thus, nature tends to insist that the greater the deviation of the rate of growth or decline from the mean, or expected growth rate, μ, the *less* likely it is to occur. As we saw in Chapter 6.6, the *normal*

[2] Equations that are referenced in subsequent chapters are in bold face.

probability distribution identifies such a process, as captured by the growth rate, μ, and standard deviation, σ, of possible growth outcomes about μ.

As observed in Chapter 6.11, the central limit theorem implies that repeated independent sampling from a distribution (with replacement) leads to a *normal distribution* of outcomes. Thus, if stock price movements occur as new information is forthcoming, with independent impacts accumulating to update the stock price in an unforeseen manner, we must anticipate that stock returns are normally distributed. The assumption that such a process represents at least an approximation of stock price movements has been justified based on the evidence of past stock price performance (for example, Fama, 1976, Chapter 2).

We must bear in mind, however, that in accepting the normal distribution as indicative of stock price growth formation, a key assumption is that each updating of the stock price is *independent* from either previous or subsequent updates. If stock prices are subject to cascades of either euphoria or pessimism, leading to self-fuelling price movements, the assumption of independent stock prices changes is violated. Studies in fact indicate that stock returns are more accurately described as exhibiting leptokurtosis, meaning that values of the exponential growth rate, x, both near to the mean and highly divergent from the mean appear more likely than predicted by a normal distribution for x (Jackwerth and Rubinstein, 1996). Such leptokurtosis is consistent with recognition that the market is prone to periods of self-generated growths and declines, with the outcome that the volatility of stock prices is unstable through time.

Nevertheless, in developing the present chapter's arguments, we shall, for the present, hold to the simplifying assumption that stock price changes are independent in time and that the volatility or standard deviation, σ, of stock returns is fixed in time. The reality of "bubbles" and "bursts" will, however, concern us in subsequent chapters.

7.3 The Mathematics of Normally Distributed Growth Rates

As observed in Eq. 6.22 (p. 92), when an asset is subject to normally distributed exponential growth rates, the unknown outcome at the end of a period can be expressed as the asset value $\$X_0$ at the commencement of the period multiplied by e^y, where $y = \mu+Z\sigma$:

The end-of-the-period outcome of investing $\$X_0$

at the commencement of the period = $\$X_0\ e^{\mu+Z\sigma}$ (6.22)

where μ represents the underlying mean or drift exponential growth rate for the period with standard deviation σ, and Z represents a stochastic (random) drawing from -∞ to +∞ with probability determined by the unit normal distribution in Fig. 6.3 (p. 85).

For such growth, we now define the key concept of return, the "exponential growth rate per period", R, by

$\$X_0e^R$ ≡ expected wealth outcome of investing $\$X_0$ for a single period (7.1)

Or, alternatively (with Eq. 6.15, p. 82) as

R **≡ ln [expected wealth outcome of investing $1 for a single period] (7.2)**

where ln represents the natural log function. If returns are distributed with drift or mean growth rate, μ, and standard deviation, σ, we might expect that R must be equal to μ. Intriguingly, however, this is not the case. In fact, we have the statistical relation[3]:

$$\boldsymbol{R = \mu + \tfrac{1}{2}\sigma^2} \qquad \textbf{(7.3)}$$

To see why Eq. 7.3 is so, consider a binomial process as represented in Fig. 6.9 (p. 120), which represents a $1 investment that can grow with equal probabilities to $e^{0.693147} = 2$ or $e^{-0.693147} = ½$. We then have (with Eq. 6.61, p. 107):

μ = (for 0.69315 and -0.69315) = 0

and (with Eqs. 6.62 and 6.63, p. 107-108):

σ (for 0.69315 and -0.69315) = 0.69315.

[3] On substituting the definitions for R (Eq. 7.2), μ, and σ, in Eq. 7.3, Eq. 7.3 may alternatively be expressed: ln[expected wealth outcome] = expected value of [ln(wealth outcome)] + ½ variance of [ln(wealth outcome)].

Since $e^{0.69315} = 2$, whereas $e^{-0.69315} = ½$, the expected outcome (of the $1 investment) is $(2+½)/2 = $1.25, which implies (Eq. 7.2): $R = \ln(1.25) = 0.2231$, or <u>22.31%</u>.

Notwithstanding that Eq. 7.3 applies to a normal distribution (as opposed to the binomial distribution of Fig. 6.9), it is instructive to apply:

$$R = \mu + ½\,\sigma^2 \tag{7.3}$$

to the above binomial example. We then have

$$R = 0 + ½\ 0.69315^2 \qquad = 0.2402, \text{ or } 24.02\%,$$

which is not such a bad estimate of the value for R in the binomial example (22.31%, above).

Alternatively, we can see how Eq. 7.3 comes about algebraically by expressing the exponential growth rate, R, that delivers the expected wealth outcome in Fig. 6.7 (p. 117) as

$$\$e^R = \$[e^{\mu+\sigma} + e^{\mu-\sigma}]/2 \tag{7.4}$$

With Eq. 6.12 (p. 78), we express Eq. 7.4 as

$$\$e^R = \$[1 + \mu + \sigma + ½(\mu+\sigma)^2 + \ldots + 1 + \mu - \sigma + ½(\mu-\sigma)^2 + \ldots] / 2$$

$$= \$(1 + \mu + ½\mu^2 + ½\sigma^2 + \ldots) \tag{7.5}$$

Allowing that the binomial representation captures the exponential process over small time intervals, the exponential growth rate, μ, and standard deviation, σ, in Eq. 7.4 relate to the exponential return, μ^*, and standard deviation, σ^*, for the actual duration as $\mu = \mu^*/N$ and $\sigma = \sigma^*/\sqrt{N}$, respectively, for large N (Eqs. 6.79 and 6.80, p. 120). Thus, in Eq. 7.5, in the limit of continuous growth rates (large N), we are justified in dropping the $½\mu^2$ term (which diminishes as $1/N^2$), while maintaining the $½\sigma^2$ term (which diminishes as $1/N$). Similarly, allowing that R also diminishes as $1/N$, we have

$$e^R \simeq 1+R \tag{7.6}$$

And, hence, in the limit of a small time interval, Eq. 7.4 is expressed:

$$1 + R = 1 + \mu + ½\sigma^2$$

which is Eq. 7.3.

7.4 The Outcome of Normally Distributed Growth Rates

Investors assess stock prices not in terms of their potential exponential growth characteristics, but in terms of their "expected periodic return", by which we mean the expected percentage increase in wealth generated over some *discrete* period (a month, a year). Such return is expressed:

expected periodic return ≡

$$[\text{expected wealth outcome of investing } X_0 \text{ for a single period - } X_0] / X_0 \quad (7.7)$$

where X_0 is the initial investment. In order to generalize the relationship between the above *periodic* return – which is to say, the "surface" return that market participants seek to measure – and the "sub-surface" *continuously applied* parameters μ and σ (which generate the surface periodic return), we combine Eqs. 7.1 and 7.7:

$$1 + \text{expected periodic return} \equiv e^{R} \quad (7.8)$$

Eq. 7.8 (with Eq. 7.3) then provides

$$\mathbf{1 + expected\ periodic\ return =}\ e^{\mu + ½\sigma^2} \quad \mathbf{(7.9)}$$

or (with Eq. 6.12)

$$\mathbf{expected\ periodic\ return} \simeq \mu + ½\sigma^2 \quad \mathbf{(7.10)}$$

Since $½\sigma^2$ is necessarily positive, Eq. 7.10 is the statement that the volatility of returns (σ) in an exponential growth process acts to *increase* the expected periodic return. This is in contra-distinction with classical Markowitz (1952) portfolio theory, which holds that random variations in a portfolio must tend to *cancel* with each outer.

7.5 Normally Distributed Growth Rates and the Small Firm Size Effect in the FF-3F Model

It is possible that the additional term $½\sigma^2$ in Eq. 7.10 adds significantly to a portfolio's returns. For example, Malkiel and Xu (1997) construct portfolios as a function of the idiosyncratic volatility (the volatility of stocks that does not relate to market movements) of the portfolio's stocks measured individually. Their average annual returns range from a low of close to 12% for the portfolio with the lowest idiosyncratic volatility (5% per month) to a high close to 19% for the portfolio with the highest idiosyncratic volatility (13% per month).

The outcome is considered by Malkiel and Xu as "surprising". This is because the conventional view holds that the idiosyncratic or non-market movement of stocks within a large portfolio of stocks should cancel out by diversification. This would be the case if discrete positive deviations from the expected mean return are equally likely as discrete negative deviations.

However, as captured by Eq. 7.10, if exponential – as opposed to discrete – returns are distributed symmetrically about a mean, μ, we do not have diversification in the sense of cancelling of overall effects. Rather, the periodic return benefits from the diversification of outcomes returns as $½\sigma^2$ where σ is the standard deviation of the individual asset outcomes in the portfolio. Thus, with Eq. 7.9, an idiosyncratic volatility of asset returns of 5% per month contributes $e^{½0.05^2} - 1$ = 0.125% to the expected monthly periodic return of a portfolio of such assets (annualized = 1.50%), while an idiosyncratic volatility of asset returns of 13% per cent per month, contributes $e^{½0.13^2} - 1$ = 0.85% to the expected monthly periodic return (annualized = 10.20%). The difference of 8.7% (10.20% - 1.50%) is actually greater than Malkiel and Xu's measured difference of 7% (19% - 12%).

Malkiel and Xu observe further that the volatility of their stocks is correlated closely with the inverse of their firm size. They note thereby that their results are consistent with the the FF-3F model of Chapter 3, which predicts that portfolios of smaller firms provide rates of return that are greater than the returns from portfolios of larger firms. Thus, the small firm size effect of the FF-3F model is, on the face of it, explained by the statistical attributes of a normal distribution. However, our interpretation of the small firm size effect is vastly different from that of Fama and French, in that Eq. 7.10 interprets the additional small firm size return *not* as the outcome of investors "setting prices" in accordance with risk-return expectations, but rather as a straightforward statistical outcome of the idiosyncratic price movements of assets themselves.

7.6 Time for Reflection: What have we learned?

We have considered asset growth as subject to normally distributed continuously compounding growth rates. With no additional assumptions, the mathematics of such growth has provided a first model of asset growth. Contradicting classical Markowitz (1952) portfolio theory, the model predicts that when assets are subject to an exponential growth process, random asset variations in a portfolio of assets act to *increase* the portfolio's expected periodic return.

The model suggests that the small firm size effect in the FF-3F model may be due to the fact that portfolios comprising higher asset volatilities are *statistically* more likely to generate a higher periodic return. The essential insight is that the outcomes are not due to investors setting prices in accordance with risk-return expectations, as is generally supposed, but rather a straightforward statistical outcome of idiosyncratic asset price volatility itself. We may note that the prediction is maintained even when continuously compounding returns are not normally distributed, provided they are symmetrically distributed about the mean.